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Resolution of high order WENO schemes for complicated flow structures

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Abstract

In this short note we address the issue of numerical resolution and efficiency of high order weighted essentially non-oscillatory (WENO) schemes for computing solutions containing both discontinuities and complex solution features, through two representative numerical examples: the double Mach reflection problem and the Rayleigh–Taylor instability problem. We conclude that for such solutions with both discontinuities and complex solution features, it is more economical in CPU time to use higher order WENO schemes to obtain comparable numerical resolution.

Keywords: WENO schemes; High order accuracy; Double Mach reflection; Rayleigh–Taylor instability

1. Introduction

In this short note we address the issue of numerical resolution and efficiency of high order weighted essentially non-oscillatory (WENO) schemes for computing solutions containing both discontinuities and complex solution features, through two representative numerical examples: the double Mach reflection problem and the Rayleigh–Taylor instability problem.

The WENO schemes we use in this paper are the fifth order finite difference version developed by Jiang and Shu in [7] and the ninth order finite difference version developed by Balsara and Shu in [1]. We will only give a very rough sketch of the algorithms and refer to [7] and [1], and also to the lecture notes [10], for most details. For a conservation laws system

$$u_t + f(u)_x + g(u)_y = 0 \tag{1.1}$$

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the conservative finite difference schemes we use approximate the point values u_{ij} at a uniform (or smoothly varying) grid (x_i, y_j) in a conservative fashion. Namely, the derivative $f(u)_x$ at (x_i, y_j) is approximated along the line $y = y_j$ by a conservative flux difference

$$f(u)_x|_{x=x_i} \approx \frac{1}{\Delta x}\left(\hat{f}_{i+1/2} - \hat{f}_{i-1/2}\right),$$

where for the fifth order WENO scheme the numerical flux $\hat{f}_{i+1/2}$ depends on 5 point values $f(u_{kj})$, $k = i-2, \ldots, i+2$, when the wind is positive (i.e., when $f'(u) \geqslant 0$ for the scalar case, or when the corresponding eigenvalue is positive for the system case with a local characteristic decomposition). This numerical flux $\hat{f}_{i+1/2}$ is written as a convex combination of three third order numerical fluxes based on three different sub-stencils of three points each, and the combination coefficients depend on a "smoothness indicator" measuring the smoothness of the solution in each stencil. The resulting scheme can be proven uniformly fifth order accurate in smooth regions including at any smooth extrema. For discontinuities the solution is essentially non-oscillatory and gives sharp shock transitions. The ninth order WENO schemes follow a similar recipe, with 9 points in the stencil and 5 sub-stencils of 5 points each. The "monotonicity preserving limiters" in [1] is *not* used in this paper. We do not observe a need to further limit the solution beyond the WENO recipe for the test cases here. Time discretization is via the third order TVD Runge–Kutta method in [11]. The CFL number is taken as 0.6 for all the runs. We have not observed significant improvement for the numerical resolution when the time step is reduced or when the order of accuracy in time is increased for these test cases.

WENO finite difference schemes are relatively easy to code and have excellent parallel efficiency. However, because of the local characteristic decomposition and the evaluation of the non-linear smooth indicators, the methods are CPU time costly, especially for the higher order versions. A natural question is whether it is worthwhile to use such high order methods. The answer to this question is problem dependent. For many problems containing only simple shocks with almost linear smooth solutions in between, such as the solutions to most Riemann problems (shock tube problems), a good second order method, such as PPM [4] or other TVD [6] methods, would be the optimal choice. However, when the solution contains both discontinuities and complex solution structures in the smooth regions, a higher order method may be more economical in CPU time, as demonstrated by the examples in this paper.

2. Two numerical examples

In this section, we use the double Mach reflection problem and the Rayleigh–Taylor instability problem as examples for problems with both discontinuities and complex solution structures to demonstrate the resolution of high order WENO schemes. Both problems are for the two dimensional Euler equations of compressible gas dynamics, namely Eq. (1.1) with

$$u = (\rho, \rho\bar{u}, \rho\bar{v}, E)^{\mathrm{T}},$$

$$f(u) = (\rho\bar{u}, \rho\bar{u}^2 + p, \rho\bar{u}\bar{v}, \bar{u}(E+p))^{\mathrm{T}},$$

$$g(u) = (\rho\bar{v}, \rho\bar{u}\bar{v}, \rho\bar{v}^2 + p, \bar{v}(E+p))^{\mathrm{T}}.$$

Here ρ is the density, $(\bar{u}, \bar{v})$ is the velocity, E is the total energy, p is the pressure, related to the total energy by $E = p/(\gamma - 1) + \frac{1}{2}\rho(\bar{u}^2 + \bar{v}^2)$ with the ratio of specific heats γ being a constant.

Fifth order and ninth order finite difference WENO schemes developed by Jiang and Shu [7], Balsara and Shu [1], with the third order TVD Runge–Kutta time discretization of Shu and Osher in [11], the Lax–Friedrichs building blocks and local characteristic decomposition are used. The resolution of the two

WENO schemes are compared. As explicit methods, these WENO schemes are easily implemented on parallel computers. All computations in this paper are performed on the IBM SP parallel computer using up to 48 processors at the Technology Center for Advanced Scientific Computing and Visualization of Brown University. The parallel efficiency is over 90% when the operation per processor is kept constant, i.e., when the number of processors increases together with a mesh refinement.

We would like to emphasize that we are using the inviscid Euler equations to compute the two test cases, hence the details of the complex solution structures due to the physical instability of contact discontinuities are related to the specific form of numerical viscosity of the scheme. There is in general no strong convergence for the numerical solutions with a grid refinement in simulating such physical instabilities using Euler equations, see for example [9] and references therein. Therefore, the specific details of the complicated structures that arise in the Euler calculations from, e.g., slip lines during a mesh refinement may be (and usually are) totally non-physical and crucially depend on the type of the numerical scheme. A physically correct calculation for these problems would have to use the Navier–Stokes equations with the real physical viscosity. However, the calculations in this paper are important to identity the size of numerical viscosities. The appearance of the small structures in the flow is a measurement of the smallness of the inherent numerical viscosities of the numerical schemes. Only when the numerical viscosity is significantly smaller than the physical viscosity can the Navier–Stokes simulation be trusted. Thus the result of this paper can be used to judge on the meshes needed for an accurate Navier–Stokes simulation with a given Reynolds number when these numerical methods are used.

2.1. Double Mach reflection

This problem was initially proposed and studied in detail by Woodward and Colella [12]. It has been used extensively in the literature as a test case for high resolution schemes, see for example the adaptive mesh finite difference calculation in [2], fixed mesh discontinuous Galerkin calculation in [3] and adaptive mesh discontinuous Galerkin calculation in [8]. The computational domain is $[0,4] \times [0,1]$, and the reflecting wall lies at the bottom of the computational domain for $\frac{1}{6} \leqslant x \leqslant 4$. Initially a right-moving Mach 10 shock is positioned at $x = \frac{1}{6}$, $y = 0$ and makes a 60° angle with the x-axis. For the bottom boundary, the exact postshock condition is imposed for the part from $x = 0$ to $x = \frac{1}{6}$, and reflective boundary condition is used for the rest. At the top boundary, the flow values are set to describe the exact motion of the Mach 10 shock. Inflow and outflow boundary conditions are used for the left and right boundaries. The unshocked fluid has a density of 1.4 and a pressure of 1. The problem was run till $t = 0.2$. The ratio of specific heats $\gamma = 1.4$. The results in $[0,3] \times [0,1]$ are displayed.

We do the mesh refinement study using three different uniform meshes: $h = \frac{1}{240}, \frac{1}{480}, \frac{1}{960}$. Here and below h denotes the uniform mesh size in x and y. The density contours with 30 equally spaced contour lines from $\rho = 1.5$ to $\rho = 22.9705$ are drawn, and in Fig. 1, we display a "blown-up" portion around the double Mach stems to see the significant resolution difference between fifth order WENO (WENO5) and ninth order WENO (WENO9). It is clear that WENO9 with $h = \frac{1}{240}$ has qualitatively the same resolution as WENO5 with $h = \frac{1}{480}$ for the fine details of the complicated structure in this "blown-up" region. The same is true for WENO9 with $h = \frac{1}{480}$ and WENO5 with $h = \frac{1}{960}$. And WENO9 gives a much better resolution for these complicated solution structures than WENO5 with same mesh size. It seems that for this problem, one can use WENO9 with only half of the mesh points in each direction as in WENO5 to obtain the same resolution for the complicated flow structure.

2.2. Rayleigh–Taylor instability

Rayleigh–Taylor Instability happens on an interface between fluids with different densities when an acceleration is directed from the heavy fluid to the light fluid. The instability has a fingering nature, with

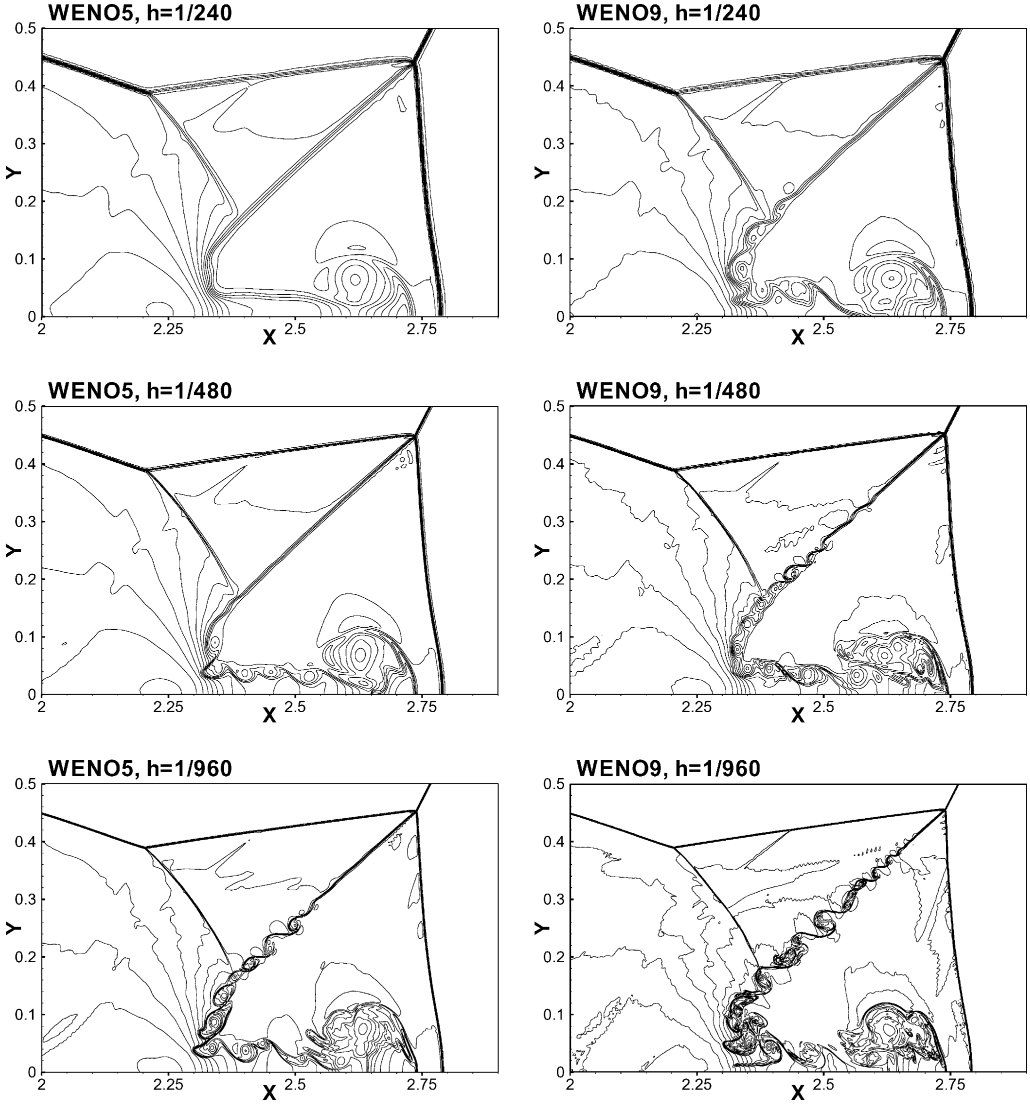

Fig. 1. Double Mach reflection problem. Blown-up region around the double Mach stems. Density ρ; 30 equally spaced contour lines from $\rho = 1.5$ to $\rho = 22.9705$. Left from top to bottom: fifth order WENO results with $h = \frac{1}{240}, \frac{1}{480}, \frac{1}{960}$; right from top to bottom: ninth order WENO results with $h = \frac{1}{240}, \frac{1}{480}, \frac{1}{960}$.

bubbles of light fluid rising into the ambient heavy fluid and spikes of heavy fluid falling into the light fluid, see for example [5] and [13]. For a discontinuous Galerkin calculation using adaptive mesh, see [8].

We set up the problem as follows: the computational domain is $[0, \frac{1}{4}] \times [0, 1]$; initially the interface is at $y = \frac{1}{2}$, the heavy fluid with density $\rho = 2$ is below the interface, and the light fluid with density $\rho = 1$ is above the interface with the acceleration in the positive y-direction; the pressure p is continuous across the interface; a small perturbation is given to the y-direction fluid speed; thus for $0 \leqslant y < \frac{1}{2}$, $\rho = 2$, $\bar{u} = 0$, $p = 2y + 1$, $\bar{v} = -0.025c \cdot \cos(8\pi x)$, and for $\frac{1}{2} \leqslant y \leqslant 1$, $\rho = 1$, $\bar{u} = 0$, $p = y + \frac{3}{2}$, $\bar{v} = -0.025c \cdot \cos(8\pi x)$, where c is the sound speed, $c = \sqrt{\frac{\gamma p}{\rho}}$, and the ratio of specific heats $\gamma = \frac{5}{3}$; reflective boundary conditions are imposed for the left and right boundaries; at the top boundary, the flow values are set as

$\rho = 1,\ p = 2.5,\ \bar{u} = \bar{v} = 0$, and at the bottom boundary, they are $\rho = 2,\ p = 1,\ \bar{u} = \bar{v} = 0$; the source terms ρ is added to the right hand side of third equation and $\rho\bar{v}$ is added to the fourth equation of Euler equations. The final simulation time is $t = 1.95$.

We do the mesh refinement study using four different uniform meshes: $h = \frac{1}{240}, \frac{1}{480}, \frac{1}{960}, \frac{1}{1920}$. The density contours with 15 equally spaced contour lines from $\rho = 0.952269$ to $\rho = 2.14589$ are shown in Fig. 2 for both WENO5 and WENO9. Again, WENO9 gives a much better resolution in complicated solution structures than WENO5 with same mesh size, and WENO9 with $h = \frac{1}{240}$ has qualitatively the same resolution as WENO5 with $h = \frac{1}{480}$, the same is true for WENO9 with $h = \frac{1}{480}$ and WENO5 with $h = \frac{1}{960}$, and for WENO9 with $h = \frac{1}{960}$ and WENO5 with $h = \frac{1}{1920}$. Thus again it seems that for this problem, one can use WENO9 with only half of the mesh points in each direction as in WENO5 to obtain the same resolution for the complicated flow structure.

For these two test cases with both discontinuities and complex solution structure, it seems that we can get comparable numerical resolution hence comparably small numerical viscosity using WENO9 with half

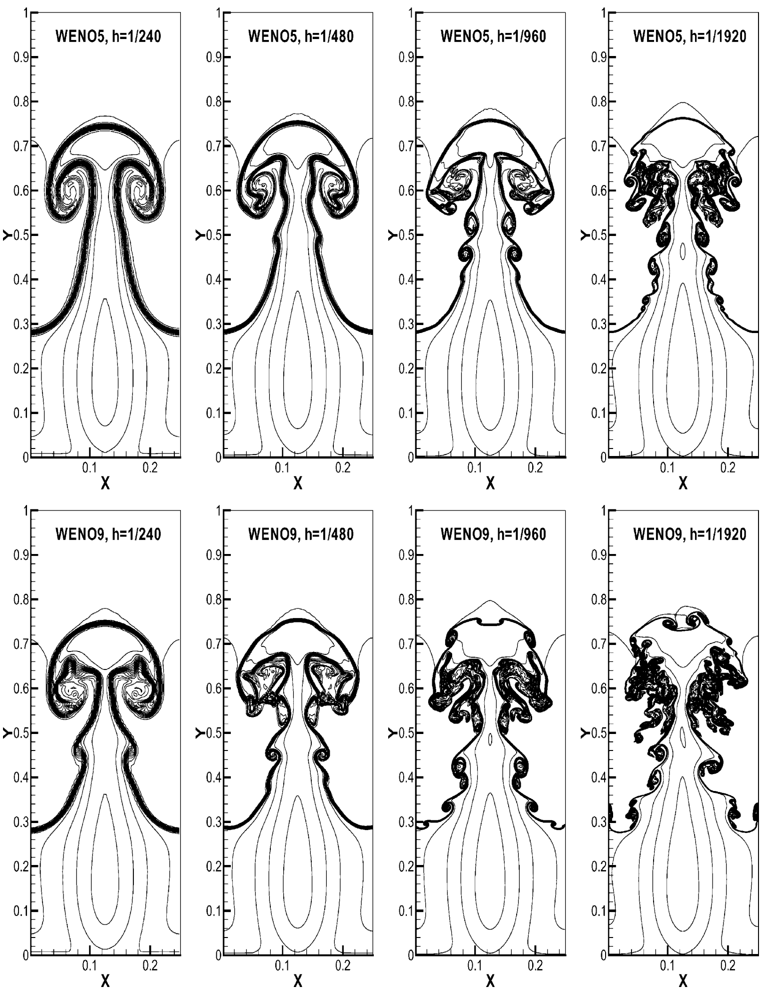

Fig. 2. Rayleigh–Taylor Instability. Density ρ; 15 equally spaced contour lines from $\rho = 0.952269$ to $\rho = 2.14589$. Top from left to right: fifth order WENO results with $h = \frac{1}{240}, \frac{1}{480}, \frac{1}{960}, \frac{1}{1920}$; bottom from left to right: ninth order WENO results with $h = \frac{1}{240}, \frac{1}{480}, \frac{1}{960}, \frac{1}{1920}$.

Table 1
Parallel CPU time for WENO5 and WENO9, unit: second, 48 processors, Rayleigh–Taylor instability problem, $t = 1.95$

h	WENO5 CPU time (s)	WENO9 CPU time (s)
1/240	168	260
1/480	780	1398
1/960	5660	9688
1/1920	47,706	75,720

Table 2
Parallel efficiency, ninth order WENO scheme, Rayleigh–Taylor instability problem, $t = 1.95$

Number of processors	$h = 1/240$		$h = 1/480$	
	CPU time (s)	Efficiency (%)	CPU time (s)	Efficiency (%)
1	5823	—	48,527	—
8	788	92.37	6111	99.26
12	565	85.88	4205	96.17
24	368	65.93	2323	87.04
48	260	46.66	1398	72.32

of the mesh points in each direction compared with the results using WENO5. This will more than offset the additional CPU cost per grid point of WENO9 versus WENO5. In Table 1, we list the CPU time for WENO5 and WENO9 using 48 processors. We can see that, when comparable resolution is obtained with WENO9 using half the number of mesh points in each direction than WENO5, the CPU time needed by WENO9 is only about 20–34% of that needed by WENO5 by our implementation. It is also important to note that this saving is more significant for more refined meshes.

Finally, in Table 2, we list the parallel efficiency for WENO9 using 8, 12, 24, 48 processors for two different mesh sizes $h = \frac{1}{240}, \frac{1}{480}$. We can see that when the per processor operation is kept constant (i.e., when the number of processors increases together with a mesh refinement), the parallel efficiency can be over 90%. The conclusion about the WENO5 scheme is the same and is not listed here to save space.

3. Concluding remarks

We have performed a numerical study on the resolution of high order WENO finite difference methods on problems containing both discontinuities and complex solution structures, using the double Mach reflection problem and the Rayleigh–Taylor instability problem as examples. We have concluded that the ninth order WENO method needs only about half the number of mesh points in each direction compared with the fifth order WENO method to achieve a comparable numerical resolution and hence to have the same small numerical viscosity. This translates to a factor of 66–80% reduction in CPU time in our implementation. Our implementation of both the fifth order and the ninth order WENO schemes are on parallel computers. The parallel efficiency can reach over 90% when the operation per processor is kept constant, i.e., when the number of mesh points increases together with the number of processors used.

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